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Notes

Broadcast time and connectivity

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Abstract

First, we give an upper bound on the broadcast time of a graph, then a new sufficient condition to have a broadcast graph. This condition will yield numerous broadcast graphs.

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1. Introduction, basic results

In this paper, we deal with the classical problem of broadcasting (see [3,7] for surveys).

We recall that the protocol is as follows:

At the step 0 some node x of a connected graph G knows a message. At the step i, any node having already received this message, may send it to one of its neighbors with the condition that all calls must use independent edges. The broadcast is completed when all nodes of G know the message.

Then the broadcast time $b(x)$ of x is the minimum number of steps necessary to complete broadcasting from x.

The broadcast time $b(G)$ of G is the greatest $b(x)$, where x is any element of the set $V(G)$ of vertices of G.

Let $v(G)$ be the order of G. We know that for every vertex x of $V(G)$, we have $b(x) \geqslant \lceil \log_2 v(G) \rceil$, and so $\lceil \log_2 v(G) \rceil$ is a lower bound on the broadcast time of G. A broadcast graph is a graph G such that $b(G) = \lceil \log_2 v(G) \rceil$.

All complete graphs are broadcast graphs, and there are broadcast graphs with fewer edges (minimum broadcast graphs). However, no general characterization of broadcast graphs is known.

In this paper, we obtain a sufficient condition for a graph to be a broadcast graph which results from a new upper bound on the broadcast time of a graph G involving the connectivity $k(G)$ of G.

Clearly, in a graph G, if we have a sequence of $r+1$ subsets $A_0, \ldots, A_r$ of $V(G)$ with $A_0 = \{x\}, A_r = V(G), A_i \subset A_{i+1}$ for any $i \in \{0, \ldots, r-1\}$, and such that for each $i \in \{0, \ldots, r-1\}$ there are exactly $|A_{i+1}| - |A_i|$ independent edges between A_i and $A_{i+1} \setminus A_i$, we can define a broadcast from x using r steps.

We also need the following known result:

Proposition 1.1. *Let G be a k-connected graph and let A be a subset of $V(G)$, with $|A| \leqslant |V(G) \setminus A|$. Then*

(a) *if $|A| \geqslant k$, there exist k independent edges between A and $V \setminus A$,*
(b) *if $|A| < k$, there exist $|A|$ independent edges between A and $V \setminus A$.*

For definitions of connectivity and k-connectivity and for a proof of Proposition 1.1, see [2].

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2. An upper bound on the broadcast time

For integers $n > 0$ and $r > 0$, we define $a_{n,r} = [\log_2 r] + 1 + \lceil (n - 2^{[\log_2 r]+1})/r \rceil$.

Proposition 2.1. *Let G be a connected graph of order v, and of connectivity k. Then $b(G) \leqslant a_{v,k}$.*

Proof. First, let us suppose that $k = 1$. In this case $a_{v,1} = v - 1$. As G is connected, for any proper non empty subset A of $V(G)$, there is a vertex in $V(G) \setminus A$ having a neighbor in A. This implies that from any $x \in V(G)$, we can construct a broadcasting scheme using $v - 1$ steps. Consequently $b(G) \leqslant v - 1$.

Now, let us suppose that $k \geqslant 2$. Let x be a vertex of G. There exists an integer m such that $2^m \leqslant k < 2^{m+1} (m = [\log_2 k])$.

For every $i \in \{0, \dots, m-1\}$ we have $2^i < k$ and $2^i \leqslant k - 2^i$ hence $2^i \leqslant v - 2^i$.

Then by Proposition 1.1, for any set A having 2^i vertices, there exist 2^i independent edges between A and $V(G) \setminus A$.

Clearly, we can construct $m + 1$ vertex sets $A_0, \dots, A_m$ such that:

- $A_0 = \{x\}$,
- $A_0 \subset \cdots \subset A_m$,
- $|A_i| = 2^i$ for every $i \in \{0, \dots, m-1\}$,
- for each $i \in \{0, \dots, m-1\}$, there are exactly $|A_{i+1}| - |A_i| = 2^i$ independent edges between A_i and $A_{i+1} \setminus A_i$.

First, suppose that $2^m \geqslant v - 2^m$.

As $2^m \leqslant k$, $v - 2^m \leqslant k$, by Proposition 1.1, there are $v - 2^m$ independent edges between A_m and $V(G) \setminus A_m$.

Therefore, there exists a broadcasting scheme from x using $m+1$ steps, and for every vertex x of G we have $b(x) \leqslant m+1$. Consequently $b(G) \leqslant m + 1$.

It is easy to verify that $-1 < (v - 2^{m+1})/k \leqslant 0$ and consequently $\lceil (v - 2^{m+1})/k \rceil = 0$.

We obtain $a_{v,k} = m + 1$ and so, the assertion holds.

Suppose now that $2^m < v - 2^m$.

Since $2^m \leqslant k$, by Proposition 1.1, there are 2^m independent edges linking A_m to $V(G) \setminus A_m$. Let A'_{m+1} be the set of end points of all these edges and let $A_{m+1} = A_m \cup A'_{m+1}$.

Then $|V(G) \setminus A_{m+1}| = v - 2^{m+1}$.

Let $s = \lceil (v - 2^{m+1})/k \rceil$.

For any vertex set A with $|A| \geqslant k$, Proposition 1.1 says that if $|V(G) \setminus A| \geqslant k$ there exist k independent edges between A and $V(G) \setminus A$, and if $|V(G) \setminus A| < k$ there exist $|V(G) \setminus A|$ independent edges between A and $V(G) \setminus A$.

This implies that we can construct s sets $A_{m+2} \subset \cdots \subset A_{m+1+s}$ with $A_{m+2} \subset \cdots \subset A_{m+1+s}, A_{m+1+s} = V(G)$ and such that for $m + 1 \leqslant i \leqslant m + s$ they are $|A_{i+1} \setminus A_i|$ independent edges between A_i and $A_{i+1} \setminus A_i$.

So, we have constructed $a_{v,k} + 1$ vertex sets $A_0, \dots, A_{m+1+s}$ defining a broadcast from x using $a_{v,k}$ steps. Therefore, for any vertex x of G we have $b(x) \leqslant a_{v,k}$ and consequently $b(G) \leqslant a_{v,k}$.

This finishes the proof of Proposition 2.1. $\square$

Remark. Proposition 2.1 implies $D(G) \leqslant a_{v,k}$ where $D(G)$ is the diameter of G.

We now give an example for which the bound of Proposition 2.1 is sharper than that of Hromkovic et al. [5].

In [5], they prove that $(\Delta(G) - 1)D(G) + 1$ is an upper bound for $b(G)$, where $\Delta(G)$ is the maximum degree of G.

For integers $r \geqslant 3$ and $k \geqslant 2$ it is easy to prove that there exists a graph $G_{r,k}$ of order rk and of connectivity k.

It is clear that $[\log_2 k] + 1 + [(rk - 2^{[\log_2 k]+1})/k] \leqslant \log_2 k + 1 + r$.

It is clear also that $(\Delta(G_{r,k}) - 1)D(G_{r,k}) + 1 \geqslant 2(k-1) + 1$. Clearly for $k \geqslant r + 2$, we have $\log_2 k + 1 + r < 2(k-1) + 1$.

This implies $a_{rk,k} < (\Delta(G_{r,k}) - 1)D(G_{r,k}) + 1$ and consequently, for the graphs $G_{r,k}$ with $k \geqslant r + 2$, $a_{rk,k}$ is a better upper bound than $(\Delta(G_{r,k}) - 1)D(G_{r,k}) + 1$.

We now show that the bound of Proposition 2.1 is sometimes sharper than that of Bermond and Peyrat concerning the undirected de Bruijn graphs $UB(d, D)$.

In [1], they prove that $b(UB(d, D)) \leqslant (d + 1)(D + 1)/2$.

It was also proved that for $D \geqslant 2$ we have $k(UB(d, D)) = 2d - 2$ (see [6]).

With our upper bound, we have $b(UB(3,2)) \leqslant 4$, $b(UB(4,2)) \leqslant 5$, $b(UB(5,2)) \leqslant 6$, $b(UB(6,2)) \leqslant 6$ and $b(UB(7,2)) \leqslant 7$.

With Bermond and Peyrat's upper bound, we have $b(UB(3,2)) \leqslant 6$, $b(UB(4,2)) \leqslant 7{,}5$, $b(UB(5,2)) \leqslant 9$, $b(UB(6,2)) \leqslant 10{,}5$ and $b(UB(7,2)) \leqslant 12$.

So, for all these graphs, our upper bound is better than the other, and it is even reached for $UB(3,2)$ and $UB(6,2)$.

However, for $D = 2$, while our upper bound is better than that of Bermond and Peyrat, it is not as good as that of Heydeman et al. [4].

3. A sufficient condition to have a broadcast graph

For integers n and r with $n \geqslant 4$ and $1 \leqslant r \leqslant n-1$, we consider the number $a_{n,r}$ already defined.

To prove the main theorem, we need several results.

Lemma 3.1. *If* $n \leqslant 2^{[\log_2 r]+1}$ *we have* $\lceil (n - 2^{[\log_2 r]+1})/r \rceil = 0$.

Proof. We have $\log_2 r \geqslant [\log_2 r]$, hence $r \geqslant 2^{[\log_2 r]}$. As $n > r$, we have $n > 2^{[\log_2 r]}$. Then $n + r > 2 \times 2^{[\log_2 r]}$, that is $n + r > 2^{[\log_2 r]+1}$, hence $(n - 2^{[\log_2 r]+1})/r > -1$ and since $(n - 2^{[\log_2 r]+1})/r \leqslant 0$, the result follows. $\square$

Let $\psi_n : \{1, \ldots, n-1\} \to N$ be the function defined by $\psi_n(r) = a_{n,r}$. Then:

Lemma 3.2. *For each* $n \geqslant 4$, ψ_n *is a decreasing function.*

Proof. For $r \in \{1, \ldots, n-2\}$, two cases are possible:

Case 1: $[\log_2 (r+1)] = [\log_2 r]$. Then we have

$\psi_n(r) = [\log_2 r] + 1 + [(n - 2^{[\log_2 r]+1})/r]$ and $\psi_n(r+1) = [\log_2 r] + 1 + [(n - 2^{[\log_2 r]+1})/(r+1)]$.

If $n - 2^{[\log_2 r]+1} > 0$, we easily deduce $\psi_n(r+1) \leqslant \psi_n(r)$.

If $n - 2^{[\log_2 r]+1} \leqslant 0$, by Lemma 3.1 we have $[(n - 2^{[\log_2 r]+1})/r] = 0$ and consequently we have $\psi_n(r+1) = \psi_n(r)$.

Case 2: $[\log_2 (r+1)] = [\log_2 r] + 1$.

Then, there exists an integer $s > 0$ such that $r + 1 = 2^s$. We get

$$\psi_n(r) = s + \left[\frac{n - 2^s}{2^s - 1}\right] = s - 1 + \left[\frac{n-1}{2^s - 1}\right],$$

$$\psi_n(r+1) = s + 1 + \left[\frac{n - 2^{s+1}}{2^s}\right] = s - 1 + \left[\frac{n}{2^s}\right].$$

As $n > 2^s$, we have $n/2^s < [(n-1)/(2^s - 1)]$ and this implies $\psi_n(r+1) \leqslant \psi_n(r)$.

In both cases we have $\psi_n(r+1) \leqslant \psi_n(r)$ and so, the assertion holds. $\square$

For the complete graph K_n, by Proposition 2.1, we have $b(K_n) \leqslant \psi_n(n-1)$, that is $\lceil \log_2 n \rceil \leqslant \psi_n(n-1)$. Consequently, for $1 \leqslant r \leqslant n-1$, we have $\lceil \log_2 n \rceil \leqslant \psi_n(r)$.

For $n \geqslant 4$ there exist unique integers m and a with $m \geqslant 1$ and $1 \leqslant a \leqslant 2^m$ such that $n = 2^m + a$. Then, we define $\theta(n)$ as follows:

$$\theta(n) = \begin{cases} 2^{m-2} + [(a+1)/2] & \text{if } 1 \leqslant a \leqslant 2^{m-1}, \\ a & \text{if } 2^{m-1} < a \leqslant 2^m. \end{cases}$$

It is easy to see that $1 \leqslant \theta(n) \leqslant n-1$ and we can state.

Proposition 3.3. *For* $\theta(n) \leqslant r \leqslant n-1$, *we have* $a_{n,r} = \lceil \log_2 n \rceil$.

For $1 \leqslant r < \theta(n)$, *we have* $a_{n,r} > \lceil \log_2 n \rceil$.

Proof. We only need to prove that $a_{n,\theta(n)} = \lceil \log_2 n \rceil$ and $a_{n,\theta(n)-1} > \lceil \log_2 n \rceil$.

Let $n = 2^m + a$ with $m \geqslant 1$ and $1 \leqslant a \leqslant 2^m$. Several cases are possible:

Case 1: $1 \leqslant a \leqslant 2^{m-1} - 2$. Then, $\lceil \log_2 n \rceil = m + 1$ and $\theta(n) = 2^{m-2} + [(a+1)/2]$.

It is easy to prove that $2^{m-2} < \theta(n) < 2^{m-1}$. Then $[\log_2 \theta(n)] = m - 2$, hence

$a_{n,\theta(n)} = m - 1 + \lceil (2^m + a - 2^{m-1})/(2^{m-2} + [(a+1)/2]) \rceil$, that is $a_{n,\theta(n)} = m - 1 + \lceil (2^{m-1} + a)/(2^{m-2} + [(a+1)/2]) \rceil$.

Clearly, $1 < (2^{m-1} + a)/(2^{m-2} + [(a+1)/2]) \leqslant 2$, and then $a_{n,\theta(n)} = \lceil \log_2 n \rceil$.

We have $\theta(n) - 1 = 2^{m-2} + [(a-1)/2]$ and then

$$a_{n,\theta(n)-1} = m - 1 + \left[\frac{2^m + a - 2^{m-1}}{2^{m-2} + [(a-1)/2]}\right] = m - 1 + \left[\frac{2^{m-1} + a}{2^{m-2} + [(a-1)/2]}\right].$$

It is easy to prove that we have $(2^{m-1}+a)/(2^{m-2}+[(a-1)/2])>2$, hence $a_{n,\theta(n)-1}>m+1$ that is $a_{n,\theta(n)-1}>\lceil\log_2 n\rceil$.

Case 2: $a=2^{m-1}-1$. Then, $\lceil\log_2 n\rceil=m+1$ and $\theta(n)=2^{m-2}+[(a+1)/2]=2^{m-1}$.

We get $a_{n,\theta(n)}=m+\lceil(2^m+2^{m-1}-1-2^m)/2^{m-1}\rceil=m+1=\lceil\log_2 n\rceil$. We also have $a_{n,\theta(n)-1}=m-1+\lceil(2^m+2^{m-1}-1-2^{m-1})/(2^{m-1}-1)\rceil=m-1+\lceil 2+1/(2^{m-1}-1)\rceil=m+2$ and then it is clear that we have $a_{n,\theta(n)-1}>\lceil\log_2 n\rceil$.

Case 3: $a=2^{m-1}$. Then, $\lceil\log_2 n\rceil=m+1$ and $\theta(n)=2^{m-2}+[(a+1)/2]=2^{m-1}$.

We get

$$a_{n,\theta(n)}=m+\left\lceil\frac{2^m+2^{m-1}-2^m}{2^{m-1}}\right\rceil=m+1=\lceil\log_2 n\rceil.$$

We also have

$a_{n,\theta(n)-1}=m-1+\lceil(2^m+2^{m-1}-2^{m-1})/(2^{m-1}-1)\rceil=m-1+\lceil 2^m/(2^{m-1}-1)\rceil$ and as $\lceil 2^m/(2^{m-1}-1)\rceil\geqslant 3$ we get $a_{n,\theta(n)-1}>\lceil\log_2 n\rceil$.

Case 4: $2^{m-1}<a<2^m$. Then, $\lceil\log_2 n\rceil=m+1$ and $\theta(n)=a$. We get

$$a_{n,\theta(n)}=m+\left\lceil\frac{2^m+a-2^m}{a}\right\rceil=m+1=\lceil\log_2 n\rceil.$$

We also have

$a_{n,\theta(n)-1}=m+\lceil(2^m+a-2^m)/a-1\rceil=m+\lceil a/(a-1)\rceil$ and since $a/(a-1)>1$ we deduce

$$a_{n,\theta(n)-1}>\lceil\log_2 n\rceil.$$

Case 5: $a=2^m$. Then, $\lceil\log_2 n\rceil=m+1$ and $\theta(n)=a=2^m$. We get

$$a_{n,\theta(n)}=m+1+\left\lceil\frac{2^{m+1}-2^{m+1}}{2^m}\right\rceil=m+1=\lceil\log_2 n\rceil.$$

We also have

$a_{n,\theta(n)-1}=m+\lceil(2^{m+1}-2^m)/(2^m-1)\rceil=m+2$ and so $a_{n,\theta(n)-1}>\lceil\log_2 n\rceil$.

So, we always have $a_{n,\theta(n)}=\lceil\log_2 n\rceil$ and $a_{n,\theta(n)-1}>\lceil\log_2 n\rceil$ and consequently the assertion is proved. □

Now we can give the main result.

Theorem 3.4. *A graph G of order v such that $k(G)\geqslant\theta(v)$ is a broadcast graph.*

Proof. By Proposition 2.1, we have $b(G)\leqslant a_{v,k(G)}$. By Proposition 3.3, we have $a_{v,k(G)}=\lceil\log_2 v\rceil$ and since $b(G)\geqslant\lceil\log_2 v\rceil$, the result follows. □

We give below a table of values of $\theta(v)$ for $4\leqslant v\leqslant 20$.

v	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\theta(v)$	2	2	2	3	4	3	3	4	4	5	6	7	8	5	5	6	6

It is clear that the smaller $\theta(v)/v$ is, the more interesting the values of $\theta(v)$ are.

The best values of $\theta(v)$ are obtained when $v=2^m+2$, with $m\geqslant 2$. Then we have $\theta(v)=2^{m-2}+1=[v/4]$, and so any graph with connectivity $\geqslant[v/4]$ yields a broadcast graph. □

Finally, we conjecture that

Conjecture 5. *For any pair (v,k) with $v\geqslant 3$ and $1\leqslant k\leqslant v-1$ there exists a graph $G_{v,k}$ of order v and of connectivity k, such that $b(G_{v,k})=a_{v,k}$.*

This conjecture is true for the pairs (v,k) *with* $k\geqslant\theta(v)$.

It is easy to prove that this conjecture is also true for the pairs $(v,1)$ ($G_{v,1}$ is a chain) and for the pairs $(v,2)$ ($G_{v,2}$ is a cycle).

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References

[1] J.-C. Bermond, C. Peyrat, Broadcasting in de Brujin networks, Proceedings of the 19th Southeastern Conference on Combinatorics, Graph theory and Computing, Congr. Numer. 66 (1988) 283–292.
[2] B. Bollobas, Extremal Graph Theory, Academic Press, New York, 1978.
[3] S.M. Hedetniemi, S.T. Hedetniemi, A.L. Liestman, A survey of gossiping and broadcasting in communication networks, Networks 18 (1988) 319–349.
[4] M.-C. Heydemann, J. Opatrny, D. Sotteau, Broadcasting and spanning trees in de Brujin and Kautz networks, Discrete Appl. Math. 27-28 (1992) 297–317.
[5] J. Hromkovic, R. Klasing, B. Monien, R. Peine, Dissemination of information in interconnection networks (broadcasting and gossiping), in: F. Hsu, D.-Z. Du (Eds.), Combinatorial Network Theory, Kluwer Academic Publishers, Dordrecht, 1995, pp. 125–212.
[6] C. Peyrat, Les réseaux d'interconnexion et leur vulnérabilité, Ph.D. Thesis, University of Paris Sud, 1987.
[7] J. de Rumeur, Communications dans les réseaux de processeurs, Masson, Paris, 1994.